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Soft Budget Constraints Problem without Interregional Transfers

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Abstract

Permanent interregional transfers are considered to be the cause of soft budget constraint (SBC) problems in local public finance. The type of SBC problem that has been analyzed in many earlier studies has the same structure as the time inconsistency problem. Therefore, if the central government can make a commitment to local governments for subsidies, the central government never revises those subsidies, local governments implement sound fiscal management, and Pareto efficiency is achieved. We show that, even if the central government can commit to a given transfer rule or there is no permanent interregional transfer system, Pareto efficiency may not be achieved in the infinite-period setting. That is, the SBC problem may still arise without interregional transfers.

JEL classification: C61; C73; H77

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1 Introduction

Permanent interregional transfers are considered to be the cause of soft budget constraint (SBC) problems in local public finance (Wildasin (1997)). However, we claim that the SBC problem may occur without a permanent interregional transfer system if the central and local governments are in an infinite-period setting. Before the Greek debt crisis, the Greek government continued to issue a large number of national bonds (Arghyrou and Tsoukalas (2011)). Consequently, the risk of default increased and finally the Greek debt crisis occurred. Note that the European Union (EU) has had no permanent interregional transfer system for a general subsidy, so our claim means that this is a type of SBC problem.

The concept of the SBC problem was originally introduced in the analyses of distortions resulting from bailouts of loss-making state-owned enterprises in a socialized economy (see Kornai (1979, 1980)), and has since been applied to the problem of subsidies provided by the central government to local governments, where it has been suggested to be a cause of distortions resulting from this redistribution.

When discussing the SBC problem in local public finance, many economists have paid particular attention to permanent intergovernmental transfers. Accordingly, they have claimed that a redistribution policy by the central government give an incentive for excessive spending. Therefore, whether or not permanent intergovernmental transfers cause the SBC problem has become an important issue. For example, Wildasin (1997) shows that an ex post bailout by the central government induces underprovision of local government expenditure for goods with a spillover effect.

The SBC problem analyzed in these earlier studies has the same structure as the time inconsistency problem. Therefore, if the central government can make a commitment to local governments for subsidies, the central government will never revise subsidies to local governments and local governments will implement sound fiscal management. However, this paper shows that this is not necessarily the case in an infinite-period model.

Some variables can be naturally introduced in a multi-period model, one of these being borrowing. This implies that it is possible to increase an excess in present consumption by overborrowing. Goodspeed (2002) shows that an ex post bailout by a central government induces overborrowing by local governments.[1]

We will show that overborrowing may arise in cases where there is no permanent interregional transfer, by extending the model to an infinite period. This will give another application of the SBC problem since the current literature presupposes the existence of a permanent interregional transfer.

This paper is organized as follows: Section 2 details a finite-period version of our model and the equilibrium conditions. Section 3 presents our main model and the equilibrium conditions. In Section 4, we examine the case that the central government can commit any transfer rule. Section 5 presents the conclusion, and some proofs are included in the appendices.

2 Finite-Period Model

In this section, we present the finite-period version of our model for comparison with the infinite-period version. The model represents a situation where, in each period, the central government decides on the amount of subsidies to be provided to the residents in each region before the local governments decide on the outstanding local bonds and the local taxes. This process is repeated a finite number of times.

[1] Wildasin (1997) and Goodspeed (2002) differ with respect to whether the fiscal scales of local governments become too large or too small with an ex post bailout by the central government. Akai and Sato (2008) indicate this difference. Caplan, Cornes and Silva (2000) show that in a built-in inefficient situation, an ex post bailout by a central government leads to Pareto efficiency because the bailout changes local governments' decisions.

The timing of the decision making of the central and local governments in the model is the opposite of that in Wildasin (1997), in which the game is not repeated. In multi-period models, this difference in timing is not critical for the SBC problem to occur.[2]

2.1 Definition of the Game

We assume a small open economy with no exogenous uncertainty. The economy consists of two regions, namely region i for $i = 1, 2$. Furthermore, the economy includes a central government and two local governments. Each region consists of a representative resident who lives for T periods.[3] The resident in region i earns one unit of income in every period and his or her preference is represented by a utility function given by

$$U^i(\{c_t^i\}_{t=1}^T, \{g_t^i\}_{t=1}^T) = \sum_{t=1}^{T} \beta^{t-1}(\ln c_t^i + \ln g_t^i),$$

where c_t^i indicates the consumption of private goods and g_t^i indicates the supply of local public goods in period t.[4] Both are assumed to be nonnegative. β is a discount factor less than one. The central government has social welfare

$$\sum_{i=1}^{2} \theta^i U^i(\{c_t^i\}_{t=1}^T, \{g_t^i\}_{t=1}^T),$$

where θ^i is positive constant for $i = 1, 2$. However, for simplicity, we assume that $\theta^1 = \theta^2 = 1$.

[2]Refer to Takahashi, Takemoto and Suzuki (2008) for details and results for the one-period and two-period versions in this section. This model is called the (finite) CL-model in Takahashi et al. (2008).

[3]Some of the literature refers to dynamic games in discrete time as "difference games," for example, de Zeeuw and van der Ploeg (1991). On the other hand, dynamic games in continuous time are called differential games. We use difference games to fairly and accurately describe the timing of the behavior of all players.

[4]The goods do not have properties such as positive externalities. We could give g_t^i the properties of public goods, but this would add unnecessary complexity to our assertion.

In the economy, the exogenous interest rate $r \in (0,1)$ is constant over time. The outstanding local bonds in period 0, x_0^i, satisfy the condition,

$$I_1(x_0^1, x_0^2) > 0,$$

where $I_t(x^1, x^2)$ is a function defined for $t = 1, \cdots, T$ as follows:

$$I_t(x^1, x^2) = 2\sum_{\tau=t}^{T}(1+r)^{t-\tau} - (1+r)(x^1 + x^2).$$

For period t, $I_t(x^1, x^2)$ represents the net discounted value of the lifetime incomes after period t in the two regions when the outstanding local bonds of local governments 1 and 2 in period $(t-1)$ are x^1 and x^2, respectively.

In period t, the players' decisions in the stage game are as follows:

First move The central government chooses the subsidy to local government 1, z_t^1, and to local government 2, z_t^2, to satisfy $z_t^1 + z_t^2 = 0$.

Second move Both local governments simultaneously choose their tax levels, y_t^i, and outstanding local bonds, x_t^i, to satisfy

$$x_t^i \geq (1+r)x_{t-1}^i - (1 + z_t^i), \tag{1}$$

$$x_T^i \leq 0, \tag{2}$$

and[5]

$$y_t^i \leq 1 + z_t^i. \tag{3}$$

Figure 1 illustrates the stage game.

After the decisions of the local governments, the consumption of private goods

[5]Equation (2) can be relaxed to $x_T^i \leq \bar{x}^i$ for arbitrary $\bar{x}^i > 0$.

and the supply of local public goods are realized satisfying the following equations:

$$c_t^i = \begin{cases} 1 - y_t^i + z_t^i & \text{if } I_t(x_{t-1}^1, x_{t-1}^2) > 0 \\ 0 & \text{otherwise} \end{cases},$$

$$g_t^i = \begin{cases} y_t^i + x_t^i - (1+r)x_{t-1}^i & \text{if } I_t(x_{t-1}^1, x_{t-1}^2) > 0 \\ 0 & \text{otherwise} \end{cases}.$$

These equations indicate that if the sum of outstanding local bonds reaches the sum of discounted future income, $\frac{2}{r}\left\{1 - \frac{1}{(1+r)^{T-t+1}}\right\}$, residents in both regions consume nothing from the next period on and redeem the bonds. This implies that if one local government has debts that are too heavy to redeem or fail to be redeemed, then the other local government bails out the residents and the creditors.

2.2 Equilibrium

Let $a_T^0 = ((s_1^1, s_1^2), (s_2^1, s_2^2), \cdots, (s_T^1, s_T^2))$ denote a strategy of the central government, where the value of s_t^i represents subsidies to the resident in region i in period t given the history up to period $(t-1)$. That is, in period t the central government decides on the subsidy amounts depending on the history up to period $(t-1)$.

Let $a_T^i = ((b_1^i, q_1^i), (b_2^i, q_2^i), \cdots, (b_T^i, q_T^i))$ denote a strategy of local government i. Here, b_i^t (respectively q_i^t) represents the outstanding local bonds (resp. local taxes) in period t given the history up to the first move of period t. In period t, the local governments decide on the outstanding local bonds and local taxes depending on both the history up to period $(t-1)$ and the central government's decision in period t.

In what follows, x_t^i (resp. y_t^i) denotes the realization of the outstanding local bonds (resp. local taxes) of local government i in period t, and z_t^i denotes the realization of the subsidy from central government to the resident in region i in

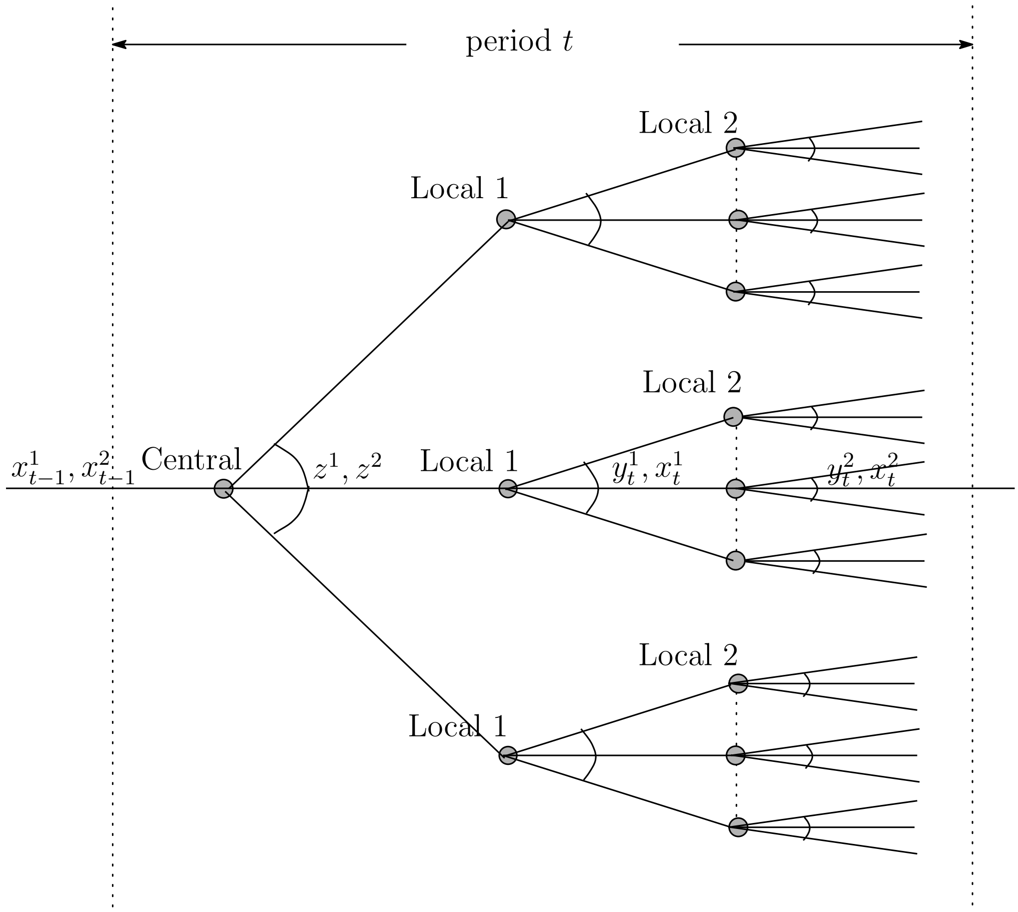

Fig. 1: Stage game

period t.

Proposition 1 If a combination of weakly undominated strategies $(a^0_T,\ a^1_T,\ a^2_T)$ is a subgame-perfect equilibrium, it satisfies the following equations for arbitrary $h = (\,\{(x^1_t, x^2_t)\}^T_{t=0},\ \{(y^1_t, y^2_t)\}^T_{t=1},\ \{(z^1_t, z^2_t)\}^T_{t=1}\,)$ such that $I_\tau(x^1_{\tau-1}, x^2_{\tau-1}) > 0$ for $\tau = 1, \cdots, T-1$:

$$s^i_T(h_{T-1}) = \frac{1}{2}(1+r)(x^i_{T-1} - x^j_{T-1}),$$

$$q^i_t(\hat{h}_t) = \begin{cases} (1+z^i_t) - \frac{1}{2}\lambda_t \cdot I_t(x^1_{t-1}, x^2_{t-1}) & \text{if } t = 1, ..., T-1 \\ \frac{1}{2}\left\{1 + z^i_T + (1+r)x^i_{T-1}\right\} & \text{if } t = T \end{cases},$$

$$b^i_t(\hat{h}_t) = \begin{cases} -(1+z^i_t) + (1+r)x^i_{t-1} + \lambda_t \cdot I_t(x^1_{t-1}, x^2_{t-1}) & \text{if } t = 1, ..., T-1 \\ 0 & \text{if } t = T \end{cases},$$

where $\lambda_t = \left(2 + \sum_{\tau=1}^{T-t} \beta^\tau\right)^{-1}$, and for $t = 1, \cdots, T-1$:

$$\hat{h}_t = (\, \{(x_\tau^1, x_\tau^2)\}_{\tau=0}^{t-1},\ \{(y_\tau^1, y_\tau^2)\}_{\tau=1}^{t-1},\ \{(z_\tau^1, z_\tau^2)\}_{\tau=1}^{t}\,),$$

and

$$h_{T-1} = (\, \{(x_\tau^1, x_\tau^2)\}_{\tau=0}^{T-1},\ \{(y_\tau^1, y_\tau^2)\}_{\tau=1}^{T-1},\ \{(z_\tau^1, z_\tau^2)\}_{\tau=1}^{T-1}\,).$$

Conversely, a combination of strategies satisfying these conditions is a subgame-perfect equilibrium.

Although there is a continuum of subgame-perfect equilibria, private and local public goods consumption paths are the same and are given as follows:

$$c_t^1 = c_t^2 = g_t^1 = g_t^2 = \frac{1}{2}\lambda_t \cdot I_t(x_{t-1}^1, x_{t-1}^2),$$

$$I_{t+1}(x_t^1, x_t^2) = (1+r)(1-\lambda_t) \cdot I_t(x_{t-1}^1, x_{t-1}^2),$$

where $\lambda_T \equiv 2^{-1}$.

The equilibrium outcome of the one-period model ($T = 1$) is efficient because $c_1^i = g_1^i$. On the other hand, if $T \geq 2$ then the equilibrium outcome is inefficient since the social planner will decide that $c_{t+1}^i = \beta(1+r)c_t^i$.[6]

2.3 Commitment of the central government

We can show that the outcome is efficient in the equilibrium of the two-player game formed by the commitment of the central government to $(\forall t \geq 1)$ $z_t^1 = z_t^2 = 0$. Furthermore, an optimal allocation of resources is achieved if the path of the subsidy satisfies $z_1^i = (1+r)(x_0^i - x_0^j)/2$ and $(\forall t \geq 2)$ $z_t^1 = z_t^2 = 0$. That is, the efficient or optimal allocation is achieved if the central government can commit to the suitable path of subsidy. Conversely, it can be said that the

[6]Refer to Takahashi et al. (2008) for the case $T = 2$.

central government's time inconsistency causes incentive problems for the local governments in the context of intergovernmental redistribution. However, these results do not hold in the infinite-period model, as we discuss in Section 4. In other words, the equilibrium outcome is not necessarily efficient or optimal in the infinite-period model even if the central government commits any transfer rule.

3 Infinite-Period Model

3.1 Definition of the Game

We assume a small open economy with no exogenous uncertainty. The economy consists of region 1 and region 2. Furthermore, the economy includes a central government and two local governments. Each region consists of a representative resident with an infinite lifespan. The resident in region i earns one unit of income in every period, and his or her preference is represented by a utility function,

$$U^i(\{c_t^i\}_{t\in\mathcal{T}}, \{g_t^i\}_{t\in\mathcal{T}}) = \sum_{t=1}^{\infty} \beta^{t-1}(\ln c_t^i + \ln g_t^i),$$

where $\mathcal{T} = \{1, 2, \cdots\}$, c_t^i indicates the consumption of private goods, and g_t^i indicates the supply of local public goods in period t. Both are assumed to be nonnegative. β is a discount factor less than one. The central government has social welfare

$$\sum_{i=1}^{2} \theta^i U^i(\{c_t^i\}_{t\in\mathcal{T}}, \{g_t^i\}_{t\in\mathcal{T}}), \tag{4}$$

where θ^i is positive constant for $i = 1,\ 2$. However, for simplicity, we assume that $\theta^1 = \theta^2 = 1$.

In the economy, the exogenous interest rate $r \in (0, 1)$ is constant over time.

The outstanding local bonds in period 0, x_0^i ($i = 1, 2$), satisfy the condition

$$I(x_0^1, x_0^2) > 0,$$

where $I(x^1, x^2)$ is a function defined as follows:

$$I(x^1, x^2) = (1+r)\left(\frac{2}{r} - x^1 - x^2\right).$$

$I(x^1, x^2)$ represents the net present value of the lifetime incomes in both regions (*gross lifetime income*), namely, the sum of today's gross income and the discounted future gross income ($2 + \frac{2}{r}$) minus the bond redemption ($(1+r)(x^1 + x^2)$).

In period t, the players' decisions in the stage game are as follows:

First move The central government chooses the subsidy to local government 1, z_t^1, and to local government 2, z_t^2, to satisfy $z_t^1 + z_t^2 = 0$.

Second move Both local governments simultaneously choose their tax levels, y_t^i, and outstanding local bonds, x_t^i, to satisfy equations (1) and (3).

Note that, unlike the finite-period model in the previous section, equation (2) is not required because there is no final period in the infinite-period model.

After the decisions of the local governments, the consumption of private goods and the supply of local public goods are realized, satisfying the following equations:

$$c_t^i = \begin{cases} 1 - y_t^i + z_t^i & \text{if } I(x_{t-1}^1, x_{t-1}^2) > 0 \\ 0 & \text{otherwise,} \end{cases} \tag{5}$$

$$g_t^i = \begin{cases} y_t^i + x_t^i - (1+r)x_{t-1}^i & \text{if } I(x_{t-1}^1, x_{t-1}^2) > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

These equations indicate that if the sum of the outstanding local bond reaches $\frac{2}{r}$, residents in both regions consume nothing from the next period on and redeem the bonds, since $x_t^1 + x_t^2 \geq \frac{2}{r}$ for all $t > \tau$ if $x_\tau^1 + x_\tau^2 \geq \frac{2}{r}$ from (1). This implies that if one local government has debts that are too heavy to redeem or fail to be redeemed, the other local government bails out the residents and the creditors.

3.2 Definition of the Equilibrium

History Let $\mathcal{B} \subset (\mathbb{R}^2)^{\mathcal{T}}$, $\mathcal{Q} \subset (\mathbb{R}^2)^{\mathcal{T}}$, and $\mathcal{S} \subset (\mathbb{R}^2)^{\mathcal{T}}$ denote the set of admissible histories for outstanding local bonds, local taxes, and subsidies from the central to local governments, respectively.

For a history of subsidies $\mathbf{z} = ((z_1^1, z_1^2), (z_2^1, z_2^2), \cdots) \in \mathcal{S}$, let $\mathbf{z}_t$ denote a history of subsidies from the first period to period t, that is, $((z_1^1, z_1^2), (z_2^1, z_2^2), \cdots, (z_t^1, z_t^2))$. Similarly, for a history of local taxes $\mathbf{y} \in \mathcal{Q}$ (resp. history of local bonds $\mathbf{x} \in \mathcal{B}$), $\mathbf{y}_t$ (resp. $\mathbf{x}_{t-1}$) denotes the first t elements of the history. For all $t \in \mathcal{T}$, the sets of these histories are denoted by $\mathcal{S}_t$, $\mathcal{Q}_t$, and $\mathcal{B}_{t-1}$. Note that $\mathcal{B}_t$ is a subset of $(\mathbb{R}^2)^{t+1}$, while $\mathcal{Q}_t$ and $\mathcal{S}_t$ are subsets of $(\mathbb{R}^2)^t$, respectively.[7]

We define the sets of histories ($\mathcal{H}$, $\mathcal{H}_t$, and $\mathcal{H}_t^F$) as follows:

$$\begin{aligned}
\mathcal{H} &\equiv \mathcal{B} \times \mathcal{Q} \times \mathcal{S}, \\
\mathcal{H}_t &\equiv \begin{cases} \mathcal{B}_0 & \text{if } t = 0 \\ \mathcal{B}_t \times \mathcal{Q}_t \times \mathcal{S}_t & \text{if } t \in \mathcal{T} \end{cases}, \\
\mathcal{H}_t^F &\equiv \begin{cases} \mathcal{B}_0 \times \mathcal{S}_1 & \text{if } t = 1 \\ \mathcal{B}_{t-1} \times \mathcal{Q}_{t-1} \times \mathcal{S}_t & \text{if } t \in \mathcal{T}/\{1\}. \end{cases}
\end{aligned}$$

Here, $\mathcal{H}_t$ denotes the set of histories up to the second move in period t, and $\mathcal{H}_t^F$, the set of histories up to the first move in period t.

[7]For formal definitions of $\mathcal{B}$, $\mathcal{Q}$, $\mathcal{S}$, $\mathcal{B}_{t-1}$, $\mathcal{Q}_t$, and $\mathcal{S}_t$, refer to Appendix A.

Strategy Set The local governments' strategy sets $\mathcal{A}^1$ and $\mathcal{A}^2$ and the central government's strategy set $\mathcal{A}^0$ are defined as follows. For $i = 1, 2$,

$$\begin{aligned}
\mathcal{A}^i \equiv & \Big\{((b_1, q_1), (b_2, q_2), \cdots) \in \Pi_{t\in\mathcal{T}} \left(\mathbb{R}^{\mathcal{H}_t^F}\right)^2 \Big| (\forall t \in \mathcal{T})(\forall h \in \mathcal{H}_t^F) \\
& 1 - q_t(h) + z_t^i \geq 0,\ q_t(h) + b_t(h) - (1 + r)x_{t-1}^i \geq 0,\ \textit{where} \\
& h = (\{(x_\tau^1, x_\tau^2)\}_{\tau=0}^{t-1},\ \{(y_\tau^1, y_\tau^2)\}_{\tau=1}^{t-1},,\ \{(z_\tau^1, z_\tau^2)\}_{\tau=1}^{t})\Big\}, \textit{and} \\
\mathcal{A}^0 \equiv & \Big\{((s_1^1, s_1^2), (s_2^1, s_2^2), \cdots) \in \Pi_{t\in\mathcal{T}} \left(\mathbb{R}^{\mathcal{H}_{t-1}}\right)^2 \Big| (\forall t \in \mathcal{T})(\forall h \in \mathcal{H}_{t-1}) \\
& s_t^1(h) + s_t^2(h) = 0\Big\}.
\end{aligned}$$

For a given $((s_1^1, s_1^2), (s_2^1, s_2^2), \cdots) \in \mathcal{A}^0$, $s_t^i \in \mathbb{R}^{\mathcal{H}_{t-1}}$ represents the subsidies to the resident in region i in period t given the history up to period $(t-1)$. On the other hand, for a given $((b_1, q_1), (b_2, q_2), \cdots) \in \mathcal{A}^i$, $b_t \in \mathbb{R}^{\mathcal{H}_t^F}$ (resp. $q_t \in \mathbb{R}^{\mathcal{H}_t^F}$) represents the outstanding local bonds (resp. local taxes) in period t given the history up to the first move of period t. In this period, the local governments decide on the outstanding local bonds and local taxes depending on both the history up to period $(t - 1)$ and the central government's decision in period t.

Definition of Equilibria Let $\mathcal{A}$ denote $\mathcal{A}^0 \times \mathcal{A}^1 \times \mathcal{A}^2$. Let a strategy $a^0 \in \mathcal{A}^0$ and a history up to period $(t - 1)$ be given. Then a^0 assigns the subsidies (z_t^1, z_t^2) in period t. Let a strategy $a^i \in \mathcal{A}^i$ $(i = 1, 2)$ and a history up to the first move in period t be given. Then a^i assigns x_t^i and y_t^i, and the consumption of private goods and the supply of local public goods are realized according to (5) and (6). Hence, when a combination of strategies $\mathbf{a} = (a^0, a^1, a^2) \in \mathcal{A}$, and a history $h \in \mathcal{H}_{t-1} \cup \mathcal{H}_t^F$ are given, the sequence of consumption in region i, $\mathbf{w}^i(\mathbf{a}, h) = (\{c_t^i\}_{t\in\mathcal{T}}, \{g_t^i\}_{t\in\mathcal{T}})$, can be determined uniquely for $i = 1, 2$.

Definition 1 A combination of strategies $(a^{0*}, a^{1*}, a^{2*}) \in \mathcal{A}$ is a subgame-perfect equilibrium of the infinite-period model if it satisfies the following conditions:

$$(\forall t \in \mathcal{T})\,(\forall h \in \mathcal{H}_{t-1})\;\; a^{0*} \in \arg\max_{a \in \mathcal{A}^0} \sum_{i=1,2} \theta^i U^i(\mathbf{w}^i((a, a^{1*}, a^{2*}), h)),$$

$$(\forall t \in \mathcal{T})\,(\forall h \in \mathcal{H}_t^F)\;\; a^{1*} \in \arg\max_{a \in \mathcal{A}^1} U^1(\mathbf{w}^1((a^{0*}, a, a^{2*}), h)),$$

$$(\forall t \in \mathcal{T})\,(\forall h \in \mathcal{H}_t^F)\;\; a^{2*} \in \arg\max_{a \in \mathcal{A}^2} U^2(\mathbf{w}^2((a^{0*}, a^{1*}, a), h)).$$

Furthermore, we define the Markov perfect equilibrium (MPE). When functions $f \in \mathbb{R}^{\mathcal{H}_0}$ and $e \in \mathbb{R}^{\mathcal{H}_1^F}$ are given, the functions $\mathbf{f} = (f_1, f_2, \cdots) \in \Pi_{t \in \mathcal{T}} \mathbb{R}^{\mathcal{H}_{t-1}}$ and $\mathbf{e} = (e_1, e_2, \cdots) \in \Pi_{t \in \mathcal{T}} \mathbb{R}^{\mathcal{H}_t^F}$ can be uniquely determined in the following manner:

$$f_1 = f \quad \text{and} \quad e_1 = e \tag{7}$$

$$(\forall t \in \mathcal{T})\; f_{t+1}(\mathbf{x}_t, \mathbf{y}_t, \mathbf{z}_t) = f(x_t^1, x_t^2), \tag{8}$$

$$(\forall t \in \mathcal{T})\; e_{t+1}(\mathbf{x}_t, \mathbf{y}_t, \mathbf{z}_{t+1}) = e(x_t^1, x_t^2, z_{t+1}^1, z_{t+1}^2), \tag{9}$$

where $\mathbf{x}_t = \{(x_\tau^1, x_\tau^2)\}_{\tau=0}^t \in \mathcal{B}_t$, $\mathbf{y}_t \in \mathcal{Q}_t$, $\mathbf{z}_t = \{(z_\tau^1, z_\tau^2)\}_{\tau=1}^t \in \mathcal{S}_t$, and $\mathbf{z}_{t+1} = \{(z_\tau^1, z_\tau^2)\}_{\tau=1}^{t+1} \in \mathcal{S}_{t+1}$. By using (7) and (8), one can construct a strategy, $a \in \mathcal{A}^0$, from a combination of functions $(s^1, s^2) \in (\mathbb{R}^{\mathcal{H}_0})^2$ if $s^1 + s^2 = 0$. Hence, we say that a combination of functions $(s^1, s^2) \in (\mathbb{R}^{\mathcal{H}_0})^2$ satisfying $s^1 + s^2 = 0$ is a Markov strategy of the central government.

Similarly, for $i = 1$, 2, one can construct a strategy, $a \in \mathcal{A}^i$, from a combination of functions $(b, q) \in (\mathbb{R}^{\mathcal{H}_1^F})^2$ by using (7) and (9), if $1 - q(h) + z^i \geq 0$ and $q(h) + b(h) - (1+r)x^i \geq 0$, where $h = (x^1, x^2, z^1, z^2)$. Hence, we say that a combination of functions $(b, q) \in (\mathbb{R}^{\mathcal{H}_0})^2$ satisfying $1 - q(h) + z^i \geq 0$ and $q(h) + b(h) - (1+r)x^i \geq 0$ is a Markov strategy of local government i.

Definition 2 A combination of Markov strategies $((s^1, s^2), (b^1, q^1), (b^2, q^2))$ is an MPE of the infinite-period model if $(a^0, a^1, a^2) \in \mathcal{A}$ is a subgame-perfect equilibrium, where a^0 is constructed from (s^1, s^2) by using (7) and (8) and, for $i = 1$, 2, a^i is constructed from (b^i, q^i) by using (7) and (9).[8]

[8]Note that the Markov restriction is not imposed on the definition of strategy sets.

The notion of time inconsistency is discussed in Kydland and Prescott (1977), and Blanchard and Fischer (1989, Chap.11) explain that a subgame-perfect equilibrium strategy of the central government is time consistent.

In the present paper, the MPE is adopted as a solution concept, and this is highly suitable for dynamic programming. We justify the use of the MPE as follows.[9] Although the folk theorem may predict Pareto efficient outcomes, some literature on public finance suggests the inefficiency of local governments, for example, Pettersson-Lidbom and Dahlberg (2005) and Doi and Ihori (2006). Undoubtedly, the outcome is unknown since the folk theorem also predicts a considerable number of other inefficient outcomes. Moreover, although every deviant player must be penalized by all the other players in any trigger strategy equilibrium (or an analogous strategy such as a carrot-and-stick strategy equilibrium), in practice no (probably local) government appears to be penalized by other governments when it raises its local taxes and/or increases its public debt.

The functional equations for the MPE, corresponding to the Bellman equations of a dynamic programming model, are as follows for $i = 1, 2$:

$$V^i(x^1, x^2) = F^i(x^1, x^2, s^1(x^1, x^2), s^2(x^1, x^2)),$$

$$F^i(h) = \ln(1 - q^i(h) + z^i) + \ln(q^i(h) + b^i(h) - (1+r)x^i) + \beta V^i(b^1(h), b^2(h)),$$

$$\sum_{i=1}^{2} V^i(x^1, x^2) = \max_{(z^1, z^2) \in \mathcal{S}_1} \sum_{i=1}^{2} F^i(x^1, x^2, z^1, z^2),$$

$$F^1(h) = \max_{x,y} \{\ln(1 - y + z^1) + \ln(y + x - (1+r)x^1) + \beta V^1(x, b^2(h))\},$$

$$F^2(h) = \max_{x,y} \{\ln(1 - y + z^2) + \ln(y + x - (1+r)x^2) + \beta V^2(b^1(h), x)\},$$

where $h = (x^1, x^2, z^1, z^2) \in \mathcal{H}_1^F$, and $((s^1, s^2), (b^1, q^1), (b^2, q^2))$ is a combination of Markov strategies. These functional equations are a necessary condition for MPE.

[9] Maskin and Tirole (2001) state that the MPE has many favorable characteristics, e.g., it requires only the coarsest of information.

3.3 Planning Problem

Before showing an MPE, in this subsection we address the problem of maximizing social welfare. This problem is as follows:

$$\max \sum_{t\,=\,1}^{\infty} \beta^{t\,-\,1} \sum_{i\,=\,1}^{2} \{\ln(1 - y_t^i + z_t^i) + \ln\left[\, y_t^i + x_t^i - (1+r)x_{t\,-\,1}^i\right]\}$$
$$\text{s.t} \quad 1 - y_t^i + z_t^i \geq 0,\ \ y_t^i + x_t^i - (1+r)x_{t-1}^i \geq 0,$$
$$z_t^1 + z_t^2 = 0;\ \ x_0^1 \text{ and } x_0^2 \text{ are given and satisfy } I(x_0^1, x_0^2) > 0.$$

From the theory of dynamic programming,[10] we can solve the problem and obtain the following result.

Proposition 2 There are values x_t^i, y_t^i, and z_t^i that solve the planning problem. The gross outstanding local bonds $x_t^1 + x_t^2$, the consumption of private goods $c_t^i = 1 - y_t^i + z_t^i$, and the supply of local public goods $g_t^i = y_t^i + x_t^i - (1+r)x_{t-1}^i$ corresponding to all the solutions are the same. Furthermore, they satisfy the following equations:

$$I(x_t^1,\, x_t^2) = \beta(1+r)I(x_{t\,-\,1}^1,\, x_{t\,-\,1}^2), \tag{10}$$

$$y_t^1 + y_t^2 = 2 - \frac{1-\beta}{2} I(x_{t-1}^1,\, x_{t-1}^2),$$
$$c_t^1 = c_t^2 = g_t^1 = g_t^2 = \frac{1-\beta}{4} I(x_{t-1}^1,\, x_{t-1}^2), \tag{11}$$
$$U^1(\{c_t^1, g_t^1\}_{t\in\mathcal{T}}) = U^2(\{c_t^2, g_t^2\}_{t\in\mathcal{T}}) = V^*(x_0^1, x_0^2),$$
$$V^*(x_0^1, x_0^2) = \frac{2}{1-\beta} \ln\left(\frac{2}{r} - x_0^1 - x_0^2\right) + \delta^*,$$
$$\delta^* = \frac{2}{(1-\beta)^2}\ \{\beta \ln \beta + (1-\beta)\ln(1-\beta) + \ln(1+r) - 2(1-\beta)\ln 2\}.$$

Equation (11) implies that (i) the consumption of private goods and the supply of local public goods are equal to each other in both regions, and (ii) they are equal

[10] Refer to Stokey, Lucas and Prescott (1989) as the standard textbook.

between regions and equal a constant fraction ($\frac{1-\beta}{4}$) of the gross lifetime income.

The social optimum path cannot be achieved as an outcome of any MPE.

Proposition 3 If a combination of Markov strategies achieves the social optimum, then it is not an MPE of the infinite-period model.

If some combination of Markov strategies achieves the social optimum and is an MPE, then $V^1 = V^2 = V^*$ and the combination of Markov strategies must satisfy the functional equations in Section 3.2. However, it is easy to see that $V^1 = V^2 = V^*$ does not satisfy the functional equations for any combination of Markov strategies.

3.4 Markov Perfect Equilibrium

Let $\hat{\mathcal{H}}_1^F = \{(x^1, x^2, z^1, z^2) \in \mathcal{H}_1^F \mid I(x^1, x^2) > 0\}$.

Proposition 4 For $i = 1, 2$, let $(\hat{b}^i, \hat{q}^i)$ be the strategy such that, for arbitrary $h \in \hat{\mathcal{H}}_1^F$,

$$\hat{q}^i(h) = 1 + z^i - \frac{1-\beta}{2(2-\beta)} I(x^1, x^2), \tag{12}$$

$$\hat{b}^i(h) = -(1 + z^i) + (1 + r)x^i + \frac{1-\beta}{2-\beta} I(x^1, x^2), \tag{13}$$

where $h = (x^1, x^2, z^1, z^2)$. Then, for an arbitrary function $s \in \mathbb{R}^{\mathcal{H}_0}$, $((\hat{s}^1, \hat{s}^2), (\hat{b}^1, \hat{q}^1), (\hat{b}^2, \hat{q}^2))$ is an MPE of the infinite-period model, where $\hat{s}^1 = -\hat{s}^2 = s$.

The functional equations in Section 3.2 are satisfied by $V^1 = V^2 = V^C$ (V^C will be presented in Corollary 2) and $((\hat{s}^1, \hat{s}^2), (\hat{b}^1, \hat{q}^1), (\hat{b}^2, \hat{q}^2))$ from Proposition 4. However, a proof for the proposition is necessary since we cannot directly apply the theory of dynamic programming (for example, Theorem 4.2 of Stokey et al. (1989)). Refer to Appendix B for the proof.

An arbitrary Markov strategy of the central government is used to construct an MPE when local government i ($i = 1, 2$) uses the Markov strategy $(\hat{b}^i, \hat{q}^i)$.

Moreover, for an arbitrary strategy $a \in \mathcal{A}^0$ of the central government, $(a, \hat{a}^1, \hat{a}^2)$ is a subgame-perfect equilibrium, where $\hat{a}^1$ and $\hat{a}^2$ are constructed from $(\hat{b}^1, \hat{q}^1)$ and $(\hat{b}^2, \hat{q}^2)$, respectively, by using (7) and (9).

Corollary 1 For an arbitrary $a \in \mathcal{A}^0$, $(a, \hat{a}^1, \hat{a}^2)$ is a subgame-perfect equilibrium, where $\hat{a}^1$ and $\hat{a}^2$ are constructed from $(\hat{b}^1, \hat{q}^1)$ and $(\hat{b}^2, \hat{q}^2)$, respectively, by using (7) and (9).

This implies that the strategies of the local governments are not controllable by the central government's strategies in the following sense. No change in strategy by the central government would provide any variation in incentives to the local governments as long as the strategies of both local governments satisfy (12) and (13). In our model, each local government has no interest in the delivery rule of the subsidies but is only concerned with the amount of those subsidies. Moreover, each local government regards these subsidies as an increase in the residents' incomes.

The local government's strategy implies that the supply of local public goods is equal to the constant rate of gross lifetime income, which allows the supply of local public goods to equal the consumption of private goods irrespective of the central government strategy.

Corollary 2 The gross outstanding local bonds, the consumption of private goods, and the supply of local public goods corresponding to x_t^i, y_t^i, and z_t^i determined by all the strategies in Proposition 4 are the same. Furthermore, they satisfy the following equations:

$$I(x_t^1, x_t^2) = \frac{\beta(1+r)}{2-\beta} I(x_{t-1}^1, x_{t-1}^2), \tag{14}$$

$$y_t^1 + y_t^2 = 2 - \frac{1-\beta}{2-\beta} I(x_{t-1}^1, x_{t-1}^2),$$

$$c_t^1 = c_t^2 = g_t^1 = g_t^2 = \frac{1-\beta}{2(2-\beta)} I(x_{t-1}^1, x_{t-1}^2), \tag{15}$$

$$U^1(\{c_t^1, g_t^1\}_{t\in\mathcal{T}}) = U^2(\{c_t^2, g_t^2\}_{t\in\mathcal{T}}) = V^C(x_0^1, x_0^2),$$

$$V^C(x_0^1, x_0^2) = \frac{2}{1-\beta}\ln\left(\frac{2}{r} - x_0^1 - x_0^2\right) + \delta^C,$$

$$\delta^C = \frac{2}{(1-\beta)^2}\left\{\beta\ln\beta + (1-\beta)\ln(1-\beta) + \ln\frac{1+r}{2-\beta} - (1-\beta)\ln 2\right\}.$$

Provided the amount of outstanding local bonds is given, the consumption of private goods in each period must equal the supply of local public goods for the temporal utility to be maximized. Under the MPE, this condition is satisfied in each period. In this sense, the intratemporal resource allocation is efficient under the MPE.

Let $\hat{x}_t$ and x_t^* denote the gross outstanding local bonds in period t corresponding to the MPE and to the optimal solution for (x_0^1, x_0^2), respectively. It can be verified that, if $x_0^1 + x_0^2 < \frac{2}{r}$, then $\hat{x}_t > x_t^*$ for all t, since $\frac{\beta(1+r)}{2-\beta} < \beta(1+r)$. In other words, the gross outstanding local bonds under this MPE are too high compared with the optimal solution. This implies overconsumption during the early periods and underconsumption in succeeding periods.

3.5 Intertemporal Inefficiency

Under the MPE, the intratemporal resource allocation is efficient in terms of the consumption of private goods being equal to the supply of the local public goods in each period. However, the outcome of the MPE is inefficient since the intertemporal resource allocation is inefficient. The reason for the distortion of intertemporal resource allocation may be explained as follows. If the interest rates are constant (r), an economic agent in the private sector who earns one unit of income in every period cannot borrow such that the outstanding amount is more than $\frac{1}{r}$. That is, the outstanding amount is constrained by the present value of future income, $\frac{1}{r}$. In contrast, in the case of local bonds backed by, or believed to be backed by, the central government, the sum of the outstanding local bonds is constrained by the discounted sum of the future incomes in both regions. In other words, the upper bound on borrowing is common between agents. For example, each agent can borrow even if the outstanding amount is greater than the discounted sum of future income, as long as the sum of the outstanding amounts is smaller than the discounted sum of the future incomes in both regions. In this case, the agents compete with each other in borrowing. This competition leads to an increase in outstanding local bonds which distorts the intertemporal resource allocation.

This borrowing competition is interpreted as follows. Between them, the local governments consume the gross lifetime income, since $x_t^1 + x_t^2 \leq \frac{2}{r}$ is equivalent to $c_t^1 + g_t^1 + c_t^2 + g_t^2 \leq I(x_{t-1}^1, x_{t-1}^2)$. By interpreting the gross lifetime income as a common resource, this game has the same structure as a social dilemma. Moreover, provided x_{t-1}^i and z_t^i are given, each region can attain an arbitrary amount of private goods c_t^i and local public goods g_t^i by setting the amount of outstanding local bonds x_t^i and local taxes y_t^i as follows:

$$x_t^i = g_t^i + c_t^i - 1 - z_t^i + (1+r)x_{t-1}^i, \quad y_t^i = 1 - c_t^i + z_t^i.$$

In other words, local governments can freely decide the amount of private goods and local public goods, as long as they satisfy $c_t^1 + g_t^1 + c_t^2 + g_t^2 \leq I(x_{t-1}^1, x_{t-1}^2)$.[11] Essentially, this is the same structure as that used in the model of Levhari and Mirman (1980).[12]

4 Open-loop Subsidy Case

In the finite-period model, if the intergovernmental subsidization system does not exist, then an efficient allocation of resources is achieved even when there is a

[11]For this reason, it is inferred that the local governments' strategies are not controllable by the central government's strategies in any subgame-perfect equilibrium.

[12]The consumption in Proposition 4 is the same as that in the MPE of the modified model of Levhari and Mirman (1980).

system to bail out the failed local government and the creditors. Furthermore, an optimal allocation of resources is achieved if the path of the subsidy satisfies $z_1^i = (1+r)(x_0^i - x_0^j)/2$ and $(\forall t \geq 2)$ $z_t^1 = z_t^2 = 0$. That is, optimal allocation is achieved if the central government can commit to the suitable path of subsidy. In this section, we show that these results do not hold in the infinite-period model. In other words, the equilibrium outcome is not necessarily efficient in the infinite-period model even if the central government commit to any transfer rule.

4.1 Definition of the Game and the Equilibrium

We set up an open-loop subsidy model that represents the situation where the central government commit to a certain path of subsidies, $\bar{\mathbf{z}} = ((\bar{z}_1^1, \bar{z}_1^2), (\bar{z}_2^1, \bar{z}_2^2), \cdots)$.

Environment The basic economic environment is the same as in the previous section, except in the following points. First, there is no central government as a player making a decision. Instead, the path of the subsidy is given and is denoted by $\bar{\mathbf{z}} = \{(\bar{z}_\tau^1, \bar{z}_\tau^2)\}_{\tau \in \mathcal{T}}$. It is assumed to satisfy, for all t and $i = 1, 2$, both $\bar{z}_t^1 + \bar{z}_t^2 = 0$ and $I_1^i(x_0^i) > 0$, where $I_t(x)$ is the function defined as follows:

$$I_t^i(x) = \sum_{\tau=0}^{\infty}(1+r)^{-\tau}(1+\bar{z}_{t+\tau}^i) - (1+r)x.$$

$I_t^i(x)$ represents the discounted lifetime income including the amount of subsidy when the amount of outstanding local bonds of region i is x at the end of period $(t-1)$.

Second, in period t, both the local governments simultaneously decide on the amount of outstanding local bonds, x_t^i, and the local taxes, y_t^i, satisfying

$$x_t^i \geq (1+r)x_{t-1}^i - (1+\bar{z}_t^i) \tag{16}$$

and

$$y_t^i \leq 1 + \bar{z}_t^i.$$

Third, (5) is replaced by

$$c_t^i = \begin{cases} 1 - y_t^i + \bar{z}_t^i & \text{if } \; I(x_{t-1}^1, x_{t-1}^2) > 0 \\ 0 & \text{otherwise} \end{cases}. \tag{17}$$

Note that z_t^i in the infinite-period model is replaced with $\bar{z}_t^i$. This environment is equivalent to the situation where the central government commits to the path of subsidy $\bar{z}_t^i$ in the model in the last section.

Strategy Set For arbitrary $t \in \mathcal{T}$, we define the sets of histories up to period $(t-1)$, $\bar{\mathcal{H}}_{t-1}$, as follows:

$$\bar{\mathcal{H}}_t \equiv \begin{cases} \mathcal{B}_0 & \text{if } \; t = 0 \\ \mathcal{B}_t \times \mathcal{Q}_t & \text{if } \; t \in \mathcal{T} \end{cases},$$

The local governments' strategy sets in the previous section are replaced with

$$\begin{aligned} \bar{\mathcal{A}}^i \;\; \equiv \;\; & \Big\{ ((b_1, q_1), (b_2, q_2), \cdots) \in \Pi_{t \in \mathcal{T}} \left(\mathbb{R}^{\bar{\mathcal{H}}_{t-1}} \right)^2 \Big| \; (\forall t \in \mathcal{T})(\forall h \in \bar{\mathcal{H}}_{t-1}) \\ & 1 - q_t(h) + \bar{z}_t^i \geq 0, \; q_t(h) + b_t(h) - (1+r)x_{t-1}^i \geq 0, \; \textit{where, for } t \geq 2, \\ & h = (\{(x_\tau^1, x_\tau^2)\}_{\tau=0}^{t-1}, \; \{(y_\tau^1, y_\tau^2)\}_{\tau=1}^{t-1}) \Big\}. \end{aligned}$$

Definition of Equilibria Let $\bar{\mathcal{A}}$ denote $\bar{\mathcal{A}}^1 \times \bar{\mathcal{A}}^2$. Let a strategy $a^i \in \bar{\mathcal{A}}^i$ and a history up to period $(t-1)$ be given. Then a^i determines x_t^i and y_t^i, and the consumption of private goods and the supply of local public goods are realized according to (17) and (6). Hence, when a combination of strategies, $\mathbf{a} = (a^1, a^2) \in \bar{\mathcal{A}}$, and a history $h \in \bar{\mathcal{H}}_t$ are given, the sequence of consumption in region i, $\mathbf{w}^i(\mathbf{a}, h) = (\{c_t^i\}_{t \in \mathcal{T}}, \{g_t^i\}_{t \in \mathcal{T}})$, can be determined uniquely for $i = 1, 2$.

Definition 3 A combination of strategies $(a^{1*}, a^{2*}) \in \bar{\mathcal{A}}$ is a subgame-perfect equilibrium of the open-loop subsidy model if it satisfies the following conditions:

$$(\forall t \in \mathcal{T})\,(\forall h \in \bar{\mathcal{H}}_{t-1}) \;\; a^{1*} \in \arg\max_{a \in \bar{\mathcal{A}}^1} U^1(\mathbf{w}^1((a, a^{2*}), h)), \tag{18}$$

$$(\forall t \in \mathcal{T})\,(\forall h \in \bar{\mathcal{H}}_{t-1}) \;\; a^{2*} \in \arg\max_{a \in \bar{\mathcal{A}}^2} U^2(\mathbf{w}^2((a^{1*}, a), h)). \tag{19}$$

4.2 Subgame-Perfect Equilibrium

For an arbitrary $t \in \mathcal{T}$, let $\tilde{\mathcal{H}}_t = \{(\{(x^1_\tau, x^2_\tau)\}^t_{\tau=0}, \{(y^1_\tau, y^2_\tau)\}^t_{\tau=1}) \in \bar{\mathcal{H}}_t \mid I(x^1_t, x^2_t) > 0\}$. Furthermore, let $\tilde{\mathcal{H}}_0 = \bar{\mathcal{H}}_0$ for descriptive purposes. The following propositions present the equilibria of the model.

Proposition 5 For $i = 1, 2$, $\bar{a}^i = ((\bar{b}^i_1, \bar{q}^i_1), (\bar{b}^i_2, \bar{q}^i_2), \cdots)$ is the strategy such that

$$(\forall t \in \mathcal{T})\,(\forall h \in \tilde{\mathcal{H}}_{t-1})\; \bar{q}^i_t(h) = 1 + \bar{z}^i_t - \frac{1-\beta}{2(2-\beta)} I(x^1_{t-1}, x^2_{t-1}), \tag{20}$$

$$(\forall t \in \mathcal{T})\,(\forall h \in \tilde{\mathcal{H}}_{t-1})\; \bar{b}^i_t(h) = -(1 + \bar{z}^i_t) + (1+r)x^i_{t-1} + \frac{1-\beta}{2-\beta} I(x^1_{t-1}, x^2_{t-1}), \tag{21}$$

where $h = (\{(x^1_\tau, x^2_\tau)\}^{t-1}_{\tau=0}, \{(y^1_\tau, y^2_\tau)\}^{t-1}_{\tau=1})$ for $t \geq 2$. Then $(\bar{a}^1, \bar{a}^2)$ is a subgame-perfect equilibrium of the open-loop subsidy model.

By substituting $\bar{z}^i_t$ into equations (12) and (13), we obtain (20) and (21). The situation where each local government chooses a strategy $\bar{a}^i$ in this model is nearly equivalent to the following. The central government commits to a path of subsidy $\bar{\mathbf{z}}$ and each local government chooses a strategy $\hat{a}^i$ in the infinite-period model.[13] As in Corollary 1, we can show that $\bar{a}^i$ is a best response to $\bar{a}^j$ for $i = 1$, 2 and $i \neq j$. Refer to Appendix C for the proof.

Equations (14) and (15) describe the consumption path in these equilibria. That is, the consumption paths in these equilibria coincide with the consumption

[13] The set of histories in this model is slightly different from that in the infinite period model.

path in the equilibria in Proposition 4. Note that this consumption path does not maximize (4) and obviously dose not attain a Pareto efficient outcome.

Proposition 6 For $i = 1, 2$, $a^{i**} = ((b_1^{i**}, q_1^{i**}), (b_2^{i**}, q_2^{i**}), \cdots)$ is the strategy such that

$$q_t^{i**}(h) = 1 + \bar{z}_t^i - \frac{1-\beta}{2}\max\{\, I_t^i(x_{t-1}^i),\, 0\,\},$$

$$b_t^{i**}(h) = \begin{cases} -(1+\bar{z}_t^i) + (1+r)x_{t-1}^i + (1-\beta)I_t^i(x_{t-1}^i) & \text{if } I_t^i(x_{t-1}^i) > 0 \\ \max\left\{\, \frac{2}{r} - (1+r)x_{t-1}^j + (1+\bar{z}_t^j),\ (1+r)x_{t-1}^i - (1+\bar{z}_t^i)\,\right\} & \text{otherwise,} \end{cases}, \tag{22}$$

for all $t \in \mathcal{T}$ and for all $h \in \tilde{\mathcal{H}}_{t-1}$, where $j \neq i$ and $h = (\{(x_\tau^1, x_\tau^2)\}_{\tau=0}^{t-1}, \{(y_\tau^1, y_\tau^2)\}_{\tau=1}^{t-1})$. Then (a^{1**}, a^{2**}) is a subgame-perfect equilibrium of the open-loop subsidy model.

Refer to Appendix D for the proof.

The equilibrium consumption path coincides with the optimal growth path in the case that the amount of outstanding local bonds in region i is $x_0^i - \sum_{s=1}^{\infty}(1+r)^{-s}\bar{z}_s^i$, the central government does not pay any subsidy, and the central government never bails out any bankrupt region and creditor.[14] In other words, it is equivalent to the optimal growth path when any region is completely isolated. Therefore, this consumption path is Pareto efficient. Note, however, that this does not automatically mean social welfare is maximized. If $\bar{\mathbf{z}} = ((\bar{z}_1^1, \bar{z}_1^2), (\bar{z}_2^1, \bar{z}_2^2), \cdots)$ satisfies $\bar{z}_1^i = (1+r)(x_0^i - x_0^j)/2$ and $(\forall t \geq 2)$ $z_t^1 = z_t^2 = 0$, the equilibrium consumption maximizes social welfare.

Suppose that region 1 chooses the strategy $\bar{a}^1$ in Proposition 5. If region 2 chooses the strategy a^{2**} in Proposition 6, the amount of outstanding local bonds in region 1 increases heavily and the sum of the amount of outstanding local bonds for both regions must reach $\frac{2}{r}$. Subsequently, both regions cannot consume anything and the present and future incomes of both regions are credited to the

[14]In the equilibrium, the consumption of private goods is equal to the supply of local public goods in each region. Equation (10) represents the transition of the gross lifetime income.

repayment of the debt of region 1. Therefore, if region 1 employs the strategy $\bar{a}^1$, the best reply of region 2 to the strategy of region 1 is to employ the strategy $\bar{a}^2$.

5 Conclusion

In this paper, we show that even if the central government can commit to any transfer rule or if there is no permanent interregional transfer system, the SBC problem may arise if the central government bails out a bankrupt region and creditors. In the finite-period model, if there is no permanent interregional transfer system, (other than weakly dominated strategies) the subgame-perfect equilibrium outcome is Pareto efficient even when the central government bails out bankrupt regions and creditors. Moreover, in the infinite-period setting, a strategy profile which achieves a Pareto efficient outcome is a subgame-perfect equilibrium.[15] However, Proposition 5 implies that there is another equilibrium in which the outcome is Pareto inefficient. Furthermore, there is no guarantee that the outcome of Proposition 6 is Pareto superior to that of Proposition 5.[16] That is, there is no guarantee that two regions can cooperate to attain the efficient equilibrium in Proposition 6. Therefore, provided there is no permanent interregional transfer system, the SBC problem may arise if the central government is able to bail out a bankrupt region and creditors.

In the finite-period model, the commitment to an open-loop subsidy by the central government makes the outcome optimal. Furthermore, in the infinite-period setting, a strategy profile which maximizes social welfare is a subgame-perfect equilibrium if the central government commits to an open-loop subsidy. However, Proposition 5 implies that there is another equilibrium in which the outcome is Pareto inefficient, and therefore does not maximize social welfare.[17]

[15] Set $\bar{\mathbf{z}} = ((0,0),(0,0),\cdots)$ in Proposition 6.

[16] If $|x_0^i - x_0^j|$ is sufficiently large, the region which has a larger amount of outstanding local bonds receives larger utilities in the consumption path Proposition 5.

[17] In this case, the outcome of Proposition 6 Pareto dominates that of Proposition 5. That

Note that there are two types of commitment the central government can make regarding bailing out local governments that should be distinguished. One is a commitment to a permanent interregional transfer system and the other is guaranteeing debts from local bonds. In this paper, we do not examine the latter. In addition, one may desire a more generalized utility function and endogenously determined interest rates. These issues are left for further studies.

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is, by some refinement of equilibria, the path which maximizes social welfare may be the only equilibrium.

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Appendices

A Definition of the History Sets

The formal definitions of $\mathcal{B}$, $\mathcal{Q}$, $\mathcal{S}$, $\mathcal{B}_{t-1}$, $\mathcal{Q}_t$, and $\mathcal{S}_t$ are as follows.

$$
\begin{aligned}
\mathcal{B} &\equiv \left\{\{(x_{t-1}^1, x_{t-1}^2)\}_{t\in\mathcal{T}} \in (\mathbb{R}^2)^{\mathcal{T}} \middle| \ (\forall t \in \mathcal{T}) \ I(x_t^1, x_t^2) \le 0 \text{ if } I(x_{t-1}^1, x_{t-1}^2) \le 0\right\},\\
\mathcal{Q} &\equiv \left\{\{(y_t^1, y_t^2)\}_{t\in\mathcal{T}} \in (\mathbb{R}^2)^{\mathcal{T}} \middle| \ (\forall t \in \mathcal{T}) \ y_t^1 + y_t^2 \le 2\right\},\\
\mathcal{S} &\equiv \left\{\{(z_t^1, z_t^2)\}_{t\in\mathcal{T}} \in (\mathbb{R}^2)^{\mathcal{T}} \middle| \middle| \ (\forall t \in \mathcal{T}) \ z_t^1 + z_t^2 = 0\right\},\\
\mathcal{B}_{t-1} &\equiv \left\{ \mathbf{x}_{t-1} \in (\mathbb{R}^2)^t \middle| \ (\exists \mathbf{x} \in \mathcal{B}) \ (\mathbf{x}_{t-1}, \mathbf{x}) \in \mathcal{B}\right\},\\
\mathcal{Q}_t &\equiv \left\{ \mathbf{y}_t \in (\mathbb{R}^2)^t \middle| \ (\exists \mathbf{y} \in \mathcal{Q}) \ (\mathbf{y}_t, \mathbf{y}) \in \mathcal{Q}\right\},\\
\mathcal{S}_t &\equiv \left\{ \mathbf{z}_t \in \left(\mathbb{R}^2\right)^t \middle| \ (\exists \mathbf{z} \in \mathcal{S}) \ (\mathbf{z}_t, \mathbf{z}) \in \mathcal{S}\right\}.
\end{aligned}
$$

B Proof of Proposition 4

To prove the proposition, it is sufficient to show that the following conditions hold because of the recursive nature of the model.

$$(\forall h \in \mathcal{H}_0) \ \ \hat{a}^0 \in \arg\max_{a\in\mathcal{A}^0} \sum_{i=1,2} U^i(\mathbf{w}^i((a, \hat{a}^1, \hat{a}^2), h)), \tag{B.1}$$

$$(\forall h \in \mathcal{H}_1^F) \ \ \hat{a}^1 \in \arg\max_{a\in\mathcal{A}^1} U^1(\mathbf{w}^1((\hat{a}^0, a, \hat{a}^2), h)), \tag{B.2}$$

$$(\forall h \in \mathcal{H}_1^F) \;\; \hat{a}^2 \in \arg\max_{a \in \mathcal{A}^2} U^2(\mathbf{w}^2((\hat{a}^0, \hat{a}^1, a), h)), \tag{B.3}$$

where $\hat{a}^0, \hat{a}^1, \hat{a}^2$ are the strategies constructed from the Markov strategies $(\hat{b}^1, \hat{q}^1)$, $(\hat{b}^2, \hat{q}^2)$, and $(\hat{s}^1, \hat{s}^2)$.

For an arbitrary combination of strategies $\mathbf{a} \in \mathcal{A}$, if $I(x^1, x^2) \leq 0$ and either $h = (x^1, x^2, z^1, z^2) \in \mathcal{H}_1^F$ or $h = (x^1, x^2) \in \mathcal{H}_0$, then $U^i(\mathbf{w}^i(\mathbf{a}, h)) = -\infty$ for $i = 1, 2$. That is, if $I(x^1, x^2) \leq 0$ and $h = (x^1, x^2) \in \mathcal{H}_0$, then condition (B.1)is met, while if $I(x^1, x^2) \leq 0$ and $h = (x^1, x^2, z^1, z^2) \in \mathcal{H}_1^F$ then conditions (B.2) and (B.3) hold. Hence, we will prove these conditions for arbitrary $h \in \mathcal{H}_0$, $h \in \mathcal{H}_1^F$ such that $I(x^1, x^2) > 0$.

Let $\hat{\mathcal{H}}_0 = \{(x^1, x^2) \in \mathcal{H}_0 \mid I(x^1, x^2) > 0\}$ and $\hat{\mathcal{H}}_1^F = \{(x^1, x^2, z^1, z^2) \in \mathcal{H}_1^F \mid I(x^1, x^2) > 0\}$. For arbitrary $h \in \hat{\mathcal{H}}_0$ and $a \in \mathcal{A}^0$, we can verify that $U^i(\mathbf{w}^i((a, \hat{a}^1, \hat{a}^2), h)) = V^C(x^1, x^2)$. This equation is equivalent to condition (B.1).

Next, we will prove condition (B.2), noting that condition (B.3) can be proved in a similar way. Let $\hat{\mathcal{A}} = \{((b_1, q_1), \cdots) \in \mathcal{A}^1 \,|\, (\forall t \in \mathcal{T}) \; b_t + \hat{b}^2 \leq \frac{2}{r}\}$, , so that obviously $\hat{a}^1 \in \hat{\mathcal{A}}$. Then $U^1(\mathbf{w}^1((\hat{a}^0, a, \hat{a}^2), h)) = -\infty$ for all $a \in \mathcal{A}^1 / \hat{\mathcal{A}}$. Hence, it is sufficient to show that the following condition is satisfied:

$$(\forall h \in \hat{\mathcal{H}}_1^F) \;\; \hat{a}^1 \in \arg\max_{a \in \hat{\mathcal{A}}} U^1(\mathbf{w}^1((\hat{a}^0, a, \hat{a}^2), h)). \tag{B.4}$$

For the remainder of this section, h denotes $(x^1, x^2, z^1, z^2) \in \hat{\mathcal{H}}_1^F$ and $\hat{\mathbf{a}}_a^{-1}$ denotes $(\hat{a}^0, a, \hat{a}^2)$ for arbitrary $a \in \hat{\mathcal{A}}$.

We define the mapping $\hat{T}^\ell$: $(\mathbb{R} \cup \{-\infty, \infty\})^{\hat{\mathcal{H}}_0} \to (\mathbb{R} \cup \{-\infty, \infty\})^{\hat{\mathcal{H}}_1^F}$ as follows:

$$\hat{T}^\ell F(h) = \sup_{(b,\,q) \in \mathcal{X}_h^C} \left\{ \ln(1 - q + z^1) + \ln\left[q + b - (1+r)x^1\right] + \beta F(b, \hat{b}^2(h)) \right\},$$

where $\mathcal{X}_h^C = \{(b, q) \in \mathbb{R}^2 \,|\, 1 - q + z^1 \geq 0,\; q + b - (1+r)x^1 \geq 0,\; (b, \hat{b}^2(h)) \in \hat{\mathcal{H}}_0\}$.

Furthermore, we define the mapping $\hat{T}^c$: $(\mathbb{R} \cup \{-\infty, \infty\})^{\hat{\mathcal{H}}_1^F} \to (\mathbb{R} \cup \{-\infty, \infty\})^{\hat{\mathcal{H}}_0}$

and the operator $\hat{T}$: $(\mathbb{R} \cup \{-\infty, \infty\})^{\hat{\mathcal{H}}_0} \to (\mathbb{R} \cup \{-\infty, \infty\})^{\hat{\mathcal{H}}_0}$ by

$$\hat{T}^c F(x^1, x^2) = \sup_{(z^1, z^2) \in \mathcal{S}_1} F(x^1, x^2, z^1, z^2) \quad \text{and} \quad \hat{T}F = \hat{T}^c \hat{T}^\ell F.$$

Lemma B.1 The operator $\hat{T}$ is monotonic, that is,

$$(\forall F,\, G \in (\mathbb{R} \cup \{-\infty, \infty\})^{\hat{\mathcal{H}}_0}) \quad F \leq G \Rightarrow \hat{T}F \leq \hat{T}G.^{18}$$

Proof Suppose that $F \leq G$ for F, $G \in (\mathbb{R} \cup \{-\infty, \infty\})^{\hat{\mathcal{H}}_0}$. Then the following inequalities hold for arbitrary $(b, q) \in \mathcal{X}_h^F$:

$$\begin{aligned} &\ln(1 - q + z^1) + \ln\{q + b - (1 + r)x^1\} + \beta F(b, \hat{b}^2(h)) \\ \leq\ &\ln(1 - q + z^1) + \ln\{q + b - (1 + r)x^1\} + \beta G(b, \hat{b}^2(h)) \\ \leq\ &\hat{T}^\ell G(x^1, x^2, z^1, z^2) \leq \hat{T}G(x^1, x^2). \end{aligned}$$

Hence, $\hat{T}^\ell F(h) \leq \hat{T}G(x^1, x^2)$.

Therefore, $\hat{T}F(x^1, x^2) = \sup_{(z^1, z^2) \in \mathcal{S}_1} \hat{T}^\ell F(x^1, x^2, z^1, z^2) \leq \hat{T}G(x^1, x^2)$. □

We define the functions $\hat{V}^C \in (\mathbb{R} \cup \{-\infty, \infty\})^{\hat{\mathcal{H}}_1^F}$, W^C, V_0, and $\bar{V}^C \in (\mathbb{R} \cup \{-\infty, \infty\})^{\hat{\mathcal{H}}_0}$ by

$$\hat{V}^C(h) = \sup_{a \in \hat{A}} U^1(\mathbf{w}^1(\hat{\mathbf{a}}_a^{-1}, h)), \quad W^C = \hat{T}^c \hat{V}^C, \quad V_0 = \frac{2}{1 - \beta} \ln\left(\frac{2}{r} - x^1 - x^2\right),$$

$$\bar{V}^C(x^1, x^2) = V_0(x^1, x^2) + \bar{\delta}^C,$$

$$\bar{\delta}^C = \frac{2}{(1 - \beta)^2} \{\beta \ln \beta + (1 - \beta) \ln(1 - \beta) + \ln(1 + r) - (1 - \beta) \ln 2\}.$$

[18] For functions f and g in $\mathbb{R}^X$, the inequality $f \leq g$ implies that $f(\mathbf{x}) \leq g(\mathbf{x})$ for all $\mathbf{x} \in X$.

Thus, by definition, the following inequality holds:

$$V^C(x^1, x^2) = U^1(\mathbf{w}^1(\hat{\mathbf{a}}, h)) \leq \hat{V}^C(h) \leq W^C(x^1, x^2). \tag{B.5}$$

Lemma B.2 $(\forall (x^1, x^2) \in \hat{\mathcal{H}}_0)$ $W^C(x^1, x^2) \leq \bar{V}^C(x^1, x^2)$.

Proof For arbitrary $a = ((b_1, q_1), (b_2, q_2), \cdots) \in \hat{\mathcal{A}}$ and arbitrary $h \in \hat{\mathcal{H}}_1^F$, let $\mathbf{w}^1(\hat{\mathbf{a}}_a^{-1}, h) = (\{c_t\}_{t\in\mathcal{T}}, \{g_t\}_{t\in\mathcal{T}})$. Furthermore, let $\{(x_{t-1}^1, x_{t-1}^2)\}_{t\in\mathcal{T}}$, $\{(y_t^1, y_t^2)\}_{t\in\mathcal{T}}$, and $\{(z_t^1, z_t^2)\}_{t\in\mathcal{T}}$ denote the sequence of outstanding local bonds, the sequence of the local tax, and the sequence of subsidies corresponding to $\hat{\mathbf{a}}_a^{-1}$ and h, respectively. Note that $x_0^i = x^i$ and $z_1^i = z^i$ for $i = 1, 2$.

For arbitrary $t \in \mathcal{T}$, since $x_t^2 = \hat{b}(x_{t-1}^1, x_{t-1}^2, z_t^1, z_t^2)$, it follows from equation (13) that

$$1 + z_t^2 + x_t^2 - (1+r)x_{t-1}^2 = \frac{1-\beta}{2-\beta} I(x_{t-1}^1, x_{t-1}^2).$$

Consequently, for arbitrary t, $\{2 + x_t^1 + x_t^2 - (1+r)(x_{t-1}^1 + x_{t-1}^2)\} - \{1 + z_t^1 + x_t^1 - (1+r)x_{t-1}^1\} = 1 + z_t^2 + x_t^2 - (1+r)x_{t-1}^2 \geq 0$. Using the last inequality, we have

$$\begin{aligned} \ln c_t^1 + \ln g_t^1 &= \ln(1 - y_t^1 + z_t^1) + \ln\{y_t^1 + x_t^1 - (1+r)x_{t-1}^1\} \\ &\leq 2\ln\{1 + z_t^1 + x_t^1 - (1+r)x_{t-1}^1\} - 2\ln 2 \\ &\leq 2\ln\{2 + x_t^1 + x_t^2 - (1+r)(x_{t-1}^1 + x_{t-1}^2)\} - 2\ln 2. \end{aligned}$$

The inequality in the second line becomes equality when $y_t^1 = \{1 + z_t^1 - x_t^1 + (1+r)x_{t-1}^1\}/2$. Hence,

$$\begin{aligned} U^1(\mathbf{w}^1(\hat{\mathbf{a}}_a^{-1}, h)) &= \sum_{t=1}^{\infty} \beta^{t-1}\{\ln c_t^i + \ln g_t^i\} \\ &\leq 2\sum_{t=1}^{\infty} \beta^{t-1} \ln\{2 + x_t^1 + x_t^2 - (1+r)(x_{t-1}^1 + x_{t-1}^2)\} - \frac{2\ln 2}{1-\beta}. \end{aligned}$$

Applying dynamic programming, for an arbitrary admissible sequence of outstanding of local bonds $((x^1, x^1_1, \cdots), (x^2, x^2_1, \cdots))$ we have

$$\begin{aligned}
&\sum_{t=1}^{\infty} \beta^{t-1} \ln\{2 + x^1_t + x^2_t - (1+r)(x^1_{t-1} + x^2_{t-1})\} \\
\leq \; & \frac{1}{1-\beta} \ln I(x^1, x^2) + \frac{1}{(1-\beta)^2}\{\beta \ln \beta + (1-\beta)\ln(1-\beta) + \beta \ln(1+r)\} \\
= \; & \frac{1}{2}\bar{V}^C(x^1, x^2) + \frac{\ln 2}{1-\beta}.
\end{aligned}$$

Accordingly, $U^1(\mathbf{w}^1(\hat{\mathbf{a}}_a^{-1}, h)) \leq \bar{V}^C(x^1, x^2)$. Since $a \in \hat{\mathcal{A}}$ is selected arbitrarily, $\hat{V}^C(x^1, x^2, z^1, z^2) \leq \bar{V}^C(x^1, x^2)$ holds for arbitrary $(x^1, x^2, z^1, z^2) \in \hat{\mathcal{H}}^F_1$. Hence, the lemma follows from (B.5). □

Lemma B.3 $\lim_{n\to\infty} \hat{T}^n \bar{V}^C(x^1, x^2) = V^C(x^1, x^2)$.

Proof We can verify that for an arbitrary constant δ, $\hat{T}(V_0 + \delta) = V_0(x^1, x^2) + \delta_0 + \beta\delta$ holds, where $\delta_0 = \frac{2}{1-\beta}\left\{\beta \ln \beta + (1-\beta)\ln(1-\beta) + \ln \frac{1+r}{2-\beta} - (1-\beta)\ln 2\right\}$. Therefore, $\hat{T}^n \bar{V}^C = \hat{T}^n(V_0 + \bar{\delta}) = V_0 + (1 + \beta + \cdots + \beta^{n-1})\delta_0 + \beta^n \bar{\delta}$. Taking the limit as n tends to infinity, we have $\lim_{n\to\infty} \hat{T}^n \bar{V}^C = V^C$. □

Lemma B.4 $W^C \leq \hat{T} W^C$.

Proof For arbitrary $a = ((b_1, q_1), (b_2, q_2), \cdots) \in \hat{\mathcal{A}}$, let $\hat{U}(h)$ and $\hat{V}^C(x^1, x^2)$ denote $U^1(\mathbf{w}^1(\hat{\mathbf{a}}_a^{-1}, h))$ and $\hat{U}(x^1, x^2, \hat{s}^1(x^1, x^2), \hat{s}^2(x^1, x^2))$, respectively. Then the following inequalities hold

$$\begin{aligned}
\hat{V}^C(x^1, x^2) &= U^1(\mathbf{w}^1(\hat{\mathbf{a}}_a^{-1}, (x^1, x^2, \hat{s}^1(x^1, x^2), \hat{s}^2(x^1, x^2)))) \\
&\leq \hat{V}^C(x^1, x^2, \hat{s}^1(x^1, x^2), \hat{s}^2(x^1, x^2)) \leq W^C(x^1, x^2). \qquad \text{(B.6)} \\
U^1(\mathbf{w}^1(\hat{\mathbf{a}}_a^{-1}, h)) &= \ln\{1 - q_1(h) + z^1\} + \ln\{q_1(h) + b_1(h) - (1+r)x^1\} \\
&\quad + \beta \hat{U}(b_1(h), b_2(h)) \\
&\leq \hat{T}^\ell \hat{U}(h) \leq \hat{T}\hat{V}^C(x^1, x^2) \leq \hat{T} W^C(x^1, x^2).
\end{aligned}$$

The last inequality follows from Lemma B.1 and inequality (B.6). Hence,

$$\hat{V}^C(x^1, x^2, z^1, z^2) = \sup_{a \in \hat{\mathcal{A}}} U^1(\mathbf{w}^1(\hat{\mathbf{a}}_a^{-1}, (x^1, x^2, z^1, z^2))) \le \hat{T} W^C(x^1, x^2).$$

Therefore, $W^C = \sup_{(z^1, z^2) \in \mathcal{S}_1} \hat{V}^C(x^1, x^2, z^1, z^2) \le \hat{T} W^C$. □

Using Lemmas B.1, B.4, and B.2 repeatedly, we obtain $W^C \le \hat{T}^n W^C \le \hat{T}^n \bar{V}^C$. Therefore, $\hat{V}^C(h) \le W^C \le \hat{T}^n \bar{V}^C$. The first inequality is obtained from (B.5). Letting n approach infinity and applying Lemma B.3 with inequality (B.5), we obtain $\hat{V}^C(h) = V^C(x^1, x^2) = U^1(\mathbf{w}^1(\hat{\mathbf{a}}, h))$. This implies condition (B.4).

C Proof of Proposition 5

First, we will prove condition (18). Let $\bar{a}^0 = \{(\bar{s}_t^1, \bar{s}_t^2)\}_{t \in \mathcal{T}} \in \mathcal{A}^0$ such that

$$(\forall t \in \mathcal{T})\, (\forall h \in H_{t-1})\, (\forall i = 1, 2)\ \bar{s}_t^i(h) = \bar{z}_t^i.$$

For given $h = (\{(x_\tau^1, x_\tau^2)\}_{\tau=0}^t, \{(y_\tau^1, y_\tau^2)\}_{\tau=1}^t) \in \bar{H}_t$, let $\tilde{h}(h)$ denote the sequence $(\{(x_\tau^1, x_\tau^2)\}_{\tau=0}^t, \{(y_\tau^1, y_\tau^2)\}_{\tau=1}^t, \{(\bar{z}_\tau^1, \bar{z}_\tau^2)\}_{\tau=1}^t)$.Note that $\tilde{h}(h) \in H_{t+1}^F$.

For given $a^i = \{a_t^i\}_{t \in \mathcal{T}} \in \bar{\mathcal{A}}^i$ $(i = 1, 2)$, define the sets of strategies of local government i in the infinite-period model, $D^i(a^i)$, as follows:

$$\begin{aligned} D^i(a^i) \ &= \ \big\{ \{\tilde{a}_t^i\}_{t \in \mathcal{T}} \in \mathcal{A}^i \ \big| \ \tilde{a}_1^i(x_0^1, x_0^2, \bar{z}_1^1, \bar{z}_1^2) = a_1^i(x_0^1, x_0^2), \\ &\qquad (\forall t \in \mathcal{T})(\forall h \in \bar{H}_t)\, \tilde{a}_{t+1}^i(\tilde{h}(h)) = a_{t+1}^i(h) \big\}. \end{aligned}$$

In particular, $\hat{a}^i \in D^i(\bar{a}^i)$ for $i = 1, 2$, where $\hat{a}^i$ is constructed from the Markov strategy $(\hat{b}^i, \hat{q}^i)$ in Proposition 4 by using (7) and (9).

Choose $(a^1, a^2) = (\{(b_t^1, q_t^1)\}_{t \in \mathcal{T}}, \{(b_t^2, q_t^2)\}_{t \in \mathcal{T}}) \in \bar{\mathcal{A}}$, t, and $h \in \bar{\mathcal{H}}_{t-1}$ arbitrarily. From the definition above, it follows for $i = 1$, 2 and $\tilde{a}^i = \{(\tilde{b}_t^i, \tilde{q}_t^i)\}_{t \in \mathcal{T}} \in D^i(a^i)$

that

$$(\forall t \in \mathcal{T})\ \tilde{b}_t^i(\tilde{h}(h)) = b_t^i(h),\ \tilde{q}_t^i(\tilde{h}(h)) = q_t^i(h).$$

Therefore, since $\bar{s}_t^i(\tilde{h}(h)) = \bar{z}_t^i$,

$$(\forall t \in \mathcal{T})\,(\forall h \in \bar{H}_{t-1})\ \mathbf{w}^i((a^1, a^2), h) = \mathbf{w}^i((\bar{a}^0, \tilde{a}^1, \tilde{a}^2), \tilde{h}(h)). \qquad \text{(C.1)}$$

Hence, for $a^1 \in \bar{\mathcal{A}}^1$, $t \in \mathcal{T}$ and $h \in \bar{H}_{t-1}$,

$$\begin{aligned} U^1(\mathbf{w}^i((a^1, \bar{a}^2), h)) &= U^i(\mathbf{w}^i((\bar{a}^0, \tilde{a}^1, \hat{a}^2), \tilde{h}(h))) \\ &\leq U^i(\mathbf{w}^i((\bar{a}^0, \hat{a}^1, \hat{a}^2), \tilde{h}(h))) \\ &= U^1(\mathbf{w}^i((\bar{a}^1, \bar{a}^2), h)), \end{aligned}$$

where $\tilde{a}^1 \in D^1(a^1)$. The first and last equalities follows from equation (C.1) and $\hat{a}^i \in D^i(\bar{a}^i)$, and the inequality in the second line follows from Corollary 1. This implies condition (18). Condition (19) can be proved in a similar way.

D Proof of Proposition 6

Suppose that $h = (\{(x_\tau^1, x_\tau^2)\}_{\tau=0}^{t-1}, \{(y_\tau^1, y_\tau^2)\}_{\tau=1}^{t-1}) \in \bar{\mathcal{H}}_{t-1}$ satisfies $I(x_{t-1}^1, x_{t-1}^2) \leq 0$. Then, for an arbitrary combination of strategies $\mathbf{a} \in \bar{\mathcal{A}}$, we have $U^i(\mathbf{w}^i(\mathbf{a}, h)) = -\infty$ for $i = 1, 2$. That is, conditions (18) and (19) are met for arbitrary t if $h \in \bar{\mathcal{H}}_{t-1}$ satisfies $I(x_{t-1}^1, x_{t-1}^2) \leq 0$. Hence, we will prove the conditions for arbitrary t and $h \in \tilde{\mathcal{H}}_{t-1}$. First, the proof of condition (18) is provided below.

Take t and $h \in \tilde{\mathcal{H}}_{t-1}$ satisfying $I_t^1(x_{t-1}^1) \leq 0$. Suppose that, for some $a^1 \in \bar{\mathcal{A}}^1$, $U^1(\mathbf{w}^1((a^1, a^{2*}), h)) > -\infty$. In this case, $c_{t+\tau-1}^1 > 0$ and $g_{t+\tau-1}^1 > 0$ for arbitrary $\tau \in \mathcal{T}$, where $\mathbf{w}^1((a^1, a^{2*}), h) = (\{c_\tau^1\}_{\tau\in\mathcal{T}}, \{g_\tau^1\}_{\tau\in\mathcal{T}})$. Hence, from (6) and (17),

$$I_t^1(x_{t-1}^1) = \sum_{\tau=1}^{\infty} (1+r)^{1-\tau}(c_{t+\tau-1}^1 + g_{t+\tau-1}^1) > 0.$$

This contradicts $I_t^1(x_{t-1}^1) \leq 0$ and $U^1(\mathbf{w}^1((a^1, a^{2*}), h)) = -\infty$ for arbitrary $a^1 \in \bar{\mathcal{A}}^1$. Therefore, for arbitrary t, condition (18) is met if $I_t^1(x_{t-1}^1) \leq 0$.

Take t and $h \in \tilde{\mathcal{H}}_{t-1}$ satisfying $I_t^1(x_{t-1}^1) > 0$ and $I_t^2(x_{t-1}^2) \leq 0$. From equation (22), we have $x_t^2 \geq \frac{2}{r} - (1+r)x_{t-1}^1 + (1+\bar{z}_t^1)$, and from equation (16) we get $x_t^1 \geq (1+r)x_{t-1}^1 - (1+\bar{z}_t^1)$. Therefore, either $x_t^1 + x_t^2 \geq \frac{2}{r}$ or $I(x_t^1, x_t^2) \leq 0$. Hence, $U^1(\mathbf{w}^1((a^1, a^{2*}), h)) = -\infty$ for arbitrary $a^1 \in \bar{\mathcal{A}}^1$. That is, for arbitrary t, condition (18) is met if $I_t^1(x_{t-1}^1) > 0$ and $I_t^2(x_{t-1}^2) \leq 0$.

Take t and $h \in \tilde{\mathcal{H}}_{t-1}$ satisfying $I_t^1(x_{t-1}^1) > 0$ and $I_t^2(x_{t-1}^2) > 0$. Since the real income in region 1 during period $(t+\tau)$ is $1 + \bar{z}_{t+\tau}^1$ for any $\tau \in \mathcal{T}$, the utility in the region after period t ($\sum_{\tau=t}^{\infty} \beta^{\tau-1}(\ln c_\tau^1 + \ln g_\tau^1)$) is less than or equal to the maximum of the following optimization problem:

$$
\begin{aligned}
\max \sum_{\tau=1}^{\infty} & \beta^{\tau-1}(\ln c_\tau + \ln g_\tau) \\
& \text{s.t} \quad c_\tau + g_\tau \leq 1 + \bar{z}_{t+\tau-1}^1, \; x_0 = x_{t-1}^1, \\
& \qquad \bar{z}_{t+\tau-1}^1 \text{ and } x_{t-1}^1 \text{ are given and satisfy } I_t^1(x_{t-1}^1) > 0.
\end{aligned}
$$

Letting $(\{c_\tau^{1*}\}_{\tau \in \mathcal{T}}, \{g_\tau^{1*}\}_{\tau \in \mathcal{T}})$ denote $\mathbf{w}^1((a^{1*}, a^{2*}), h)$, it is easy to verify that $(\{c_{t+\tau}^{1*}\}_{\tau \in \mathcal{T}}, \{g_{t+\tau}^{1*}\}_{\tau \in \mathcal{T}})$ is the solution to the problem. Hence, condition (18) follows. Condition (19) can be proved in a similar way.